Notes on How to Numerically Calculate the Maximum Lyapunov Exponent

These notes, primarily for my own reference, briefly describe how my programs calculate the maximum Lyapunov exponent. There is nothing extraordinary in how I do it – the method is described in ample detail by well-known references, e.g. Wolf et al. (1985, *Physica* D **16**, 285) and Benettin et al. (1976, *Phys. Rev.* A **14**, 2338).

Consider two orbits, a "reference" orbit and a "test" orbit, separated at time t_0 by a small phase space distance d_0. We will use the test orbit as a means of calculating the value of the maximum Lyapunov exponent. Under evolution of the equations of motion, the two orbits may (or may not) separate. If the motion is chaotic, the orbits will, by definition, separate at an exponential rate. The maximum Lyapunov exponent λ is a measure of this rate of separation:

$$\lambda = \lim_{t\to\infty} \frac{1}{t-t_0} \ln \frac{d(t)}{d_0} \tag{1}$$

Hence, in the limit of infinite time,

$$\lambda = \lim_{t\to\infty} \frac{1}{t-t_0} \ln \frac{d(t)}{d_0} \tag{2}$$

In practice, we cannot afford the luxury of infinitely long integrations, so we instead calculate the instantaneous maximum Lyapunov exponent

$$\lambda(t) = \frac{1}{t-t_0} \ln \frac{d(t)}{d_0} \tag{3}$$

and, ideally, wait long enough for λ(t) to settle to approximately its asymptotic value, if indeed it is non zero for the orbit of interest. A simple method of calculating λ(t) is shown in section 1. Another practical problem is that, for chaotic orbits, the distance between reference and test particles, d(t), quickly saturates. Hence we must periodically renormalize the orbit separation. This is shown in section 2. Section 3 presents the derivation of eq. (7) in more detail.

1. Exponent Calculation

We will leave the reference orbit alone and rescale the test orbit whenever the separation d(t) has passed beyond a threshold value D. It is important that D be set small enough that it is still in the linear regime (i.e., the regime in which the linearized equations of motion are an accurate description). Define a rescaling parameter:

$$\alpha_1 \equiv \frac{d(t_1)}{d(t_0)} \tag{4}$$

where t_1 is the time at which $d(t) \geq D$. Then we can write

$$\lambda_1 = \frac{1}{t_1-t_0} \ln \frac{d_1}{d_0} = \frac{1}{t_1-t_0} \ln \alpha_1 \tag{5}$$

where $\lambda_i \equiv \lambda(t_i)$ and $d_i \equiv d(t_i)$. At this point, the test orbit is then rescaled, as shown in section 2. Similarly, for successive threshold crossings and subsequent rescalings, we have

$$
\begin{aligned}
\lambda_2 &= \frac{1}{t_2 - t_0} \ln \frac{d_2 \cdot \alpha_1}{d_0} \\
&= \frac{1}{t_2 - t_0} \ln(\alpha_1 \cdot \alpha_2) \\
\\
\lambda_3 &= \frac{1}{t_3 - t_0} \ln \frac{d_3 \cdot \alpha_2 \cdot \alpha_1}{d_0} \\
&= \frac{1}{t_3 - t_0} \ln(\alpha_1 \cdot \alpha_2 \cdot \alpha_3) \\
&\vdots
\end{aligned}
\tag{6}
$$

and so on. The multiplicative factors $\alpha_1, \alpha_1 \cdot \alpha_2, \cdots$ are derived in section 3, in case it is not intuitively obvious. We therefore conclude that the instantaneous Lyapunov exponent is

$$\boxed{\lambda_n = \frac{1}{t_n - t_0} \sum_{i=1}^{n} \ln \alpha_i} \tag{7}$$

where we have defined

$$\alpha_i \equiv \frac{d(t_i)}{d(t_0)} \tag{8}$$

As long as the rescalings take place in the linear regime, this construction is valid. Notice that, in a computer, only the accumulating sum of the natural log of the α_i need be stored. In addition, the time intervals need not be evenly spaced.

2. Renormalization of the Test Orbit.

The rescaling of the test particle orbit is performed on the test - reference phase space distance vector. Whenever the distance d(t) becomes greater than or equal to the threshold D, we scale the test particle distance from the reference particle by the factor $1/\alpha_i$, maintaining the current relative orientation between the two particles in phase space. Write the reference and test particle phase space vectors as

$$\vec{\mathbf{R}} \equiv \begin{pmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{pmatrix}_{ref} \quad \text{and} \quad \vec{\mathbf{r}} \equiv \begin{pmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{pmatrix}_{test} \tag{9}$$

Define $\vec{\rho} \equiv \vec{\mathbf{r}} - \vec{\mathbf{R}}$. Then the adjustment to the test particle phase space coordinates at time t_i is

$$\vec{\mathbf{r}}_i \leftarrow \vec{\mathbf{R}}_i + \frac{\vec{\rho}_i}{\alpha_i} \tag{10}$$

Alternatively, one could write the equivalent expression

$$\vec{\mathbf{r}}_i \leftarrow \vec{\mathbf{r}}_i - \frac{\alpha_i - 1}{\alpha_i} \cdot \vec{\rho}_i \tag{11}$$

Eq. (10) is slightly less expensive to calculate than eq. (11). All we are doing is rescaling the distance d(t),

$$d(t_i) \leftarrow \frac{d(t_i)}{\alpha_i} \tag{12}$$

in the appropriate direction in phase space.

I have found that $d_0 = 10^{-6}$ and $D = 10^{-4}$ work well in practice. The figure below shows the instantaneous Lyapunov exponent for a chaotic restricted three-body orbit, with several values of d_0 ranging from 10^{-5} to 10^{-15}, with a rescaling threshold of 10^{-4}. One can see that values of d_0 in the range 10^{-5} to 10^{-8} are adequate. Smaller than this invites numerical trouble due to the finite word size of the machine.

I have also found that any difference between using the full phase space distance

$$\sqrt{x^2+y^2+z^2+v_x^2+v_y^2+v_z^2} \qquad (13)$$

and using only the configuration space distance

$$\sqrt{x^2+y^2+z^2} \qquad (14)$$

is indiscernible.

3. Explanation of summation

In this section, for completeness, I derive the multiplicative factors in the distances in the logarithms of eq. (7) and the equations leading up to it. Consider eq. (1). At a time t_1,

$$d_1 = d_0\, e^{\lambda_1 t_1} \qquad (15)$$

Upon rescaling,

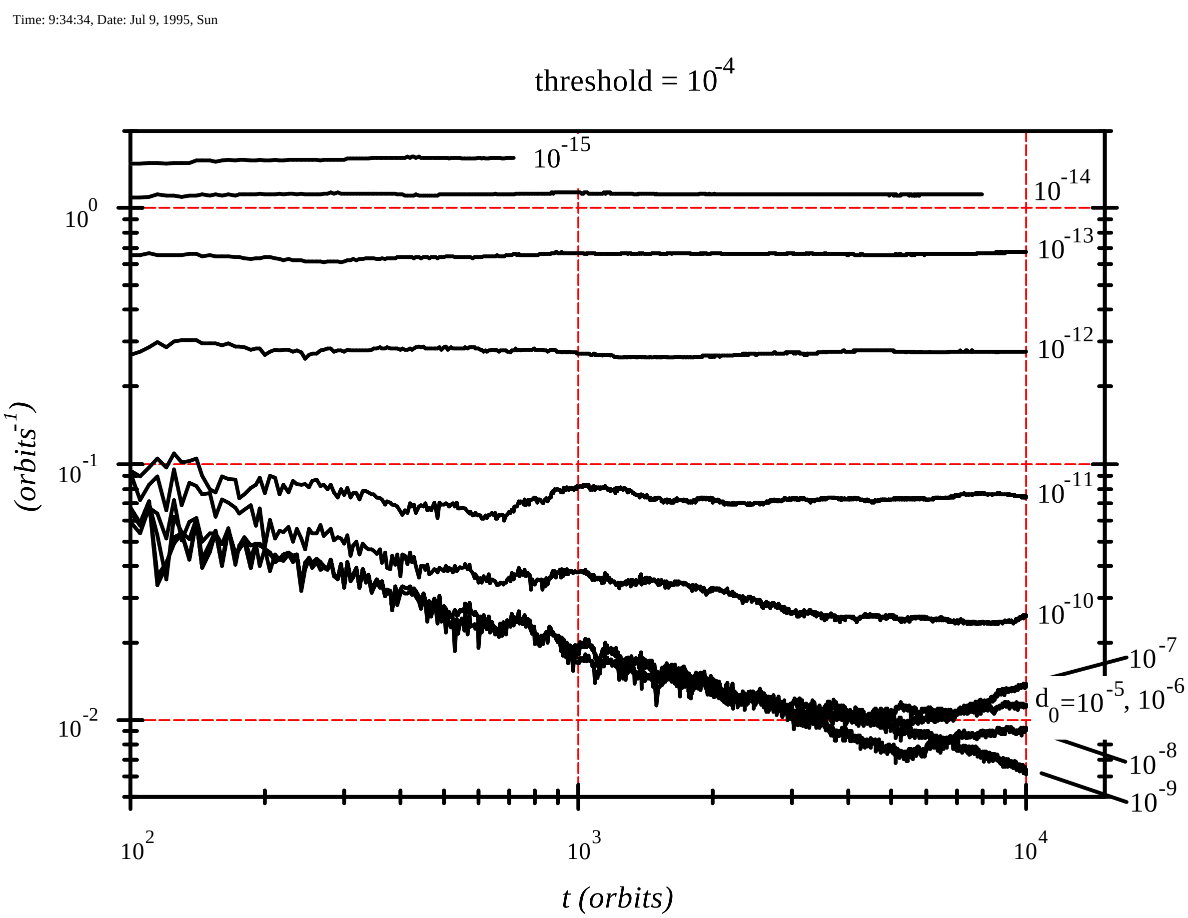
$$d_1 \leftarrow \frac{d_0 e^{\lambda_1 t_1}}{\alpha_1} \qquad (16)$$

At the next rescaling time t_2,

$$d_2 = d_1\, e^{\lambda_1 \cdot (t_2 - t_1)} \qquad (17)$$

where d_1 is the rescaled value (i.e., d_0). Inserting eq. (16) for d_1, we have

$$d_2 = \frac{d_0 e^{\lambda_2 t_2}}{\alpha_1} \qquad (18)$$



where we have assumed the increment in time is small so that $\lambda_1 \approx \lambda_2$. Upon rescaling, eq. (18) becomes

$$d_2 \leftarrow \frac{d_2}{\alpha_2} \tag{19}$$

where, using eq. (18),

$$\alpha_2 = \frac{d_2}{d_0} = \frac{e^{\lambda_2 t_2}}{\alpha_1} \tag{20}$$

Hence, from eq. (20), we have

$$\lambda_2 = \frac{1}{t_2} \ln(\alpha_1 \alpha_2) \tag{21}$$

which is eq. (6). Extending this process further, we conclude eq. (7).

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